The RAMS Analyses in the Face of Ageing.
The Bayes Approach.

C.A. Clarotti,

ENEA, Rome Italy.

F.Billy & A. Lannoy

EDF, Chatou France

1 Introduction

The RAMS analyses are thought to be an engineering based tool whose use entails no theoretical lucubration. To the contrary, many theoretical aspects must be taken into account for RAMS studies to do indeed the job they are supposed to do that is, for RAMS studies to state in a sound way whether or not a given system is "safe/reliable enough".

As a matter of fact practical RAMS exercises cope with systems whose components are costly and fail rarely notwithstanding possible aging, which is kept under control by maintenance. For such a type of components proper life tests are infeasible due to cost and duration. The reliability of these components must then be assessed by the data collected at the plants where the components are operating. These data are referred to as "field data" and are gathered under non-homogeneous stopping rules. Indeed, the different plants, which are supplying field data, may have been started up at different times in the past. The data coming from any individual plant are equivalent to data observed under a type I life test, the truncation time being the time since the plant was started up. The life test corresponding to the ensemble of plants under observation, is a combination of type I tests of different durations. **This fact, as it will be shown in section 2, makes classical statistics unsuitable for inference from field data.**

Since field data are to be processed by Bayes statistics, the role of prior knowledge is put into the open. **Prior knowledge plays a central role both in Bayesian and in classical statistics** (see section 3). In classical statistics, prior knowledge is a hidden though fundamental ingredient. The Bayesians, who officially recognize the role of prior knowledge, have the advantage and the duty to handle prior information in a rational way. This is not an easy task in the case of aging components as there is no family of conjugate priors for the parameters of aging distributions. Two complications arise due to the lack of any conjugate family, namely:

1. numerical methods must be used for computing the integrals which appear in the Bayes formulas;
2. it becomes more difficult to balance the impacts that the observed data and the prior have on the results of the inference.

The issue of assessing priors in the case of aging is addressed in section 4.

2 Field Data and the Unavoidability of Bayes Statistics.

Three steps will be needed to prove that classical statistics is unsuitable if field data are of concern. For ease of understanding, each step will be gone in an ad-hoc subparagraph.

2.1 Classical Statistics Cares also of "What Might Have Been"

Suppose we are to measure some unknown physical quantity. We have at our disposal two different instruments, the error of the *i*-th instrument is σ_I, i=1,2. Both errors are acceptable, so we decide to toss a fair coin with the stipulation that the result "head" will cause the instrument #1 to be used. We toss the coin, and it obtains head. We are orthodox frequentist and we want that in the long run the sample standard deviation will be the same as the standard deviation foreseen by error theory of classical statistics. Which error shall we attach to the measurement we made, σ_1 or rather $(\sigma_1+\sigma_2)/2$? The correct answer is the second one. So classical statistics takes into account also what might have been (it might have obtained "tail").

2.2 The impact of Stopping Rules in Classical Statistics

A type I test is terminated at a pre-established time t_0. A type II test is stopped as soon the k-th failure occurs, k being stated in advance.

Consider the Maximum Likelihood Estimator (MLE) of the unknown mean life of an exponential distribution. This estimator is the ratio between the random value of the observed total time on test and the observed number of failures. The latter quantity is random in a type I test, and is a constant in a type II test.

We note that the domain of variability of the MLE changes according to the type of test. In type II test the value of the MLE ranges from 0 (k components fail as soon as the test is started) up to $+\infty$ (the k-th failure never occurs). As opposite, in a type I test the upper bound of the value of the MLE is equal to nx t_0, where n is the number of the on test components. The upper bound is obtained when none of the on test components fails by t_0.

Imagine you want to assess by the MLE, the 90% interval estimate of the unknown parameter of an exponential distribution. You then carry out a type I and a type II test and both give rise to the same observed value of the MLE. Are the 90% interval estimates produced by the two tests the same? The right answer is: "no". This is so because to forecast the future (to assess the interval estimate) classical statistics does not concentrate on what was (the observed data) but rather it takes into account also what might have been (i.e. the non coincident variability domains of the MLE in the two types of test).

By the same token, the interval estimate obtained under the sampling plan of field data is different from the interval estimates obtained under a type I or under a type II test which produced the same data as those gathered by observing components at the plants where they are operating.

2.3 Quantifying the Impact of the Stopping Rules.

In the case of a type II test, the probability distribution of the MLE is the user-friendly $\chi(2)$ distribution.

If a type I test is of concern [1] the distribution of the MLE is an involved linear combination of $n+(n+1)n/2$ terms. Each term is an unhandy truncated $\chi(2)$ distribution.

The sampling plan of field data is so complicate, that only the Laplace transform of MLE density is available [1].

In view of the above, interval estimation is impossible in the frame of classical inference from field data.

3. Prior Knowledge and Classical Statistics

Let $x_1,\ldots,x_n$ be a sample from the Gauss distribution with unknown mean μ and known standard deviation σ. It is well known that in the long run 96% of time it will be:

$$\frac{1}{n}\sum_{i=1}^{n} x_i - 2\sigma < \mu < \frac{1}{n}\sum_{i=1}^{n} x_i + 2\sigma \tag{1}$$

There are infinitely many interval estimates of μ at the 96% confidence level. All of them are probabilistically equivalent to one another. So should we chose the estimate

$$-\infty < \mu < \frac{1}{n}\sum_{i=1}^{n} x_i + 1.69\sigma \tag{2}$$

in the long run we would be right as 96% of time, as well. Indeed the integral of the standard Gauss density over the interval (-2,+2) is equal to the integral of the standard Gauss density over the unbounded domain (-∞,+1.69); the value of both integrals is 0.96.

It is not surprising that there is an infinite number of interval estimates corresponding to the same confidence level. The value of the latter supplies the analyst with one equation, while two unknowns (the bounds of the interval) must be specified.

Wald [2] and De Finetti [3] showed that when an orthodox statistician selects one of the probabilistically equivalent interval estimates, he simply makes a Bayesian estimate by using blindly one of the priors that can be defined on the variability domain of the unknown parameter. Prior knowledge does then have an important role in classical statistics.

4. Prior Knowledge in the Face of Aging

4.1 The Statement of the Problem

Prior knowledge cannot be expelled from statistics (no matter whether Bayesian or classical), one can at the most minimize its impact on the results of inference. Cifarelli and Regazzini showed [4] that among all the priors on some unknown parameter which have the same values of the expectation and variance, the one which minimizes the impact of the non statistical knowledge is the prior belonging to the conjugate family of priors, **if such a family exists**. No conjugate family exists for the unknown parameters of any aging distribution [5, chapt..]. In this case: sensitivity analysis is the only referee in the match plaid by the prior knowledge vs. the observed data for the leadership in the inference; numerical integration must be used for computing the integrals of Bayes formulas. We will show how these two difficulties can be handled, with reference to doubly censored data from the Weibull distribution. The likelihood function at hand will be the a more involved version of the function $L(D|\ \eta,\theta)$ [6] defined hereafter . The latter likelihood function applies when one observed one life of length t, one survival up to time t_s, and m failures in the time interval (t_1,t_2). The possible more involved versions of $L(D|\ \eta,\theta)$, contain many specimens of each of the sample factors appearing in eq (3). The variability domains of the unknown parameters η and θ are respectively $(0,+\infty)$ and $(1,+\infty)$. The related Bayes integrals are then to be computed over these domains.

$$L(D|\eta,\theta)\propto\theta\left(\frac{t}{\eta}\right)^{\theta+1}\exp\left(-\left(\frac{t}{\eta}\right)^{\theta}\right)\exp\left(-\left(\frac{t_s}{\eta}\right)^{\theta}\right)\left(\exp\left(-\left(\frac{t_1}{\eta}\right)^{\theta}\right)-\exp\left(-\left(\frac{t_2}{\eta}\right)^{\theta}\right)\right)^{m} \tag{3}$$

Should $L(D|\ \eta,\theta)$ (or any more involved version of it) be the only factor appearing in the Bayes integrals, the latter could be numerically calculated [6] by applying the variable transform (4) which maps the original integration domain onto the unit square without introducing any singularity into the transformed integrand [7].

$$u=\frac{1}{\theta},\qquad v=\exp\left(-\frac{1}{\eta}\right) \tag{4}$$

Consider the pdf's:

$$f(\eta)\propto\left(\frac{a}{\eta}\right)^{b+1}\exp\left(-\left(\frac{a}{\eta}\right)^{b}\right),\quad g(\theta)\propto\left(1-\frac{1}{\theta}\right)^{\alpha-1}\left(\frac{1}{\theta}\right)^{\beta+1} \tag{5}$$

The functions arrived at by applying the variable transform (4) to the pdf's (5), are continuous on the unit square. In view of this, if $f(\eta)g(\theta)$ is the joint prior on the unknown parameters of a Weibull distribution, no numerical problem arises in applying the Bayes theorem to doubly censored Weibull data [7].

4.2 Sensitivity Analysis-Suitability of the Proposed Priors.

Any univariate distribution is uniquely defined by the set of all its moments [5, chapt…]. The latter are infinite in number, while in practice the analyst's prior knowledge is never expressed by more than two moments, namely: the expectation and the standard deviation. In other words, the analyst can simply express his one opinion about the "general trend" of a univariate prior, the analytical form of the latter being immaterial in-so-far-as the selected family of pdf's can represent a large variety of degrees of dispersion [8]. A joint pdf on two unknown parameters is uniquely defined by the moments of the marginals, plus the mixed moments [5, chapt….]. Analysts are incapable of assessing even the simplest of the mixed moments, i.e. the covariance. The product of marginals is then a joint prior matched to the non-statistical knowledge analysts can express [8]. In the light of the previous discussion, the pdf's (5) are "suitable" for performing a sensitivity analysis and eventually assessing a joint prior on η and θ., because they allow the analyst to choose in the "very rich menu" of priors described in the next paragraph.

4.3 The Sensitivity Analysis in Practice

We will describe a graphically supported procedure for assessing the prior on the unknown parameters of the Weibull distribution. The procedure was implemented in the computer code IBW3 (Inférence Bayesienne à partir de la loi de data Weibull, release 3) [8], which makes Bayes inference from doubly censored Weibull data. The assessed joint prior is proper , but some first order or second moment of its might not exist. Since first order moments and second order ones are the current deliverables of inference, IBW3 is equipped with test subprograms which ascertain whether or not the posterior possesses the moments possibly missing a priori. In order to avoid abnormal ends, the missing posterior moments are not computed. The assessment procedure is interactive and is substantially the same for the two unknown parameters. It will be described with reference to η.

1. The analyst states a confidence level c_0 and two bounds $\eta_1<\eta_2$.
2. It is set $c_i= c_0+0.05i$, i=-2, -1, 0, 1, 2.
3. The codes:
   singles out the pdf $f(\eta)$ for which (η_1,η_2) is the shortest- c_i%-confidence interval;
   computes the related existing moments up to the order 2.
4. The 5 assessed priors are represented on the same graph; the moments of these priors are reproduced on an aside-table.

By examining the graph and the summary of the moments, the analyst can decide either to stick to the assessed prior or to enter a new value of c_0 for the sake of obtaining a more (less) dispersed pdf.

The prior eventually assessed by the above procedure can be:

a. A proper pdf with neither first order nor second order moment.
b. A pdf whose expectation takes any of the possible values in the variability domains of the two parameters ($0<\eta<+\infty$, $1<\theta<+\infty$); for both parameters and for any value of the expectation, the ratio between the standard deviation and the

expectation itself can range in an interval which, for all practical purposes, is equivalent to the unbounded interval $(0,+\infty)$

5. Concluding Remarks

The discussion we have been entertaining so far, showed that practical RAMS studies must be supported by quite a bit of theory. The need for theory comes from the involved structure of the data used in RAMS analyses. The thorough understanding of the role of Bayesian methods in the RAMS frame was possible thanks to the research that ENEA and EDF have been jointly carrying out in the last fifteen years.

References

[1] D.J. Barthollomew: The Sampling Distribution of an Estimate Arising in Life Testing, Technometrics, Vol. 5, N.3, August 1963, pp 361-374.
[2] A. Wald: Statistical Decision Functions, Wiley & Sons, New York, 1950
[3] B. de Finetti: Probability, Induction and Statistics, Wiley & Sons, London, 1972.
[4] D. M. Cifarelli, E. Ragazzini: Priors for Exponential Families which Minimize the Association between Past and Future Observations, in “Probability and Bayesian Statistics”, R. Viertl Ed, Plenum Publishing Corporation, New York, 1987, pp 83-95.
[5] C. A. Clarotti:: Les techniques fréquentielles et bayesiennes au service de l’ingénieurs de Sûreté de Fonctionnement, Rapport final du Projet 8/96, Paris, June 1998.
[6] C.A Clarotti, H. Procaccia, B. Villan: Bayesian Analysis of General Failure Data. Advances in Numerical Methods, in “Probabilistic Safety Assessment and Management”, C. Cacciabue & I.A. Papazoglou Eds, Springer-Verlag, Heidelberg, 1996, pp 1289-1294
[7] P.J. Davis, P. Rabinowitz: Methods for Numerical Integration, 2nd Edition, Academic Press, New York, 1984.
[8] F. Billy, C.A. Clarotti, A. Lannoy: Modélisation de lois a priori propres pour l’inférence d’une loi de Weibull : considérations théoriques, Rapport EDF HT-56/03/007/A, Chatou, September 2003 (in French).