



Info-gap robustness analysis applied to hybrid structural reliability

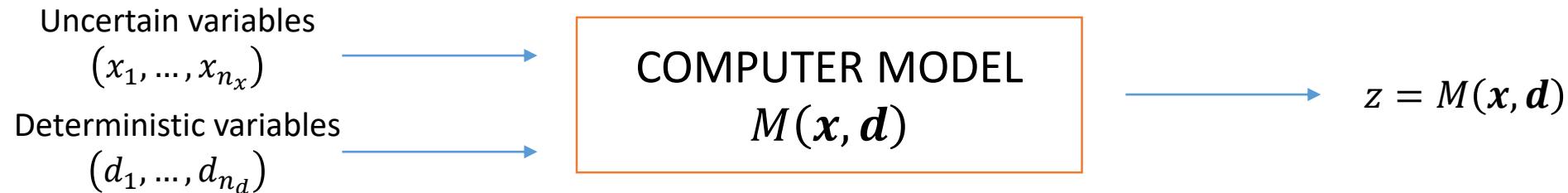
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Séminaire IMdR – 17/03/2021

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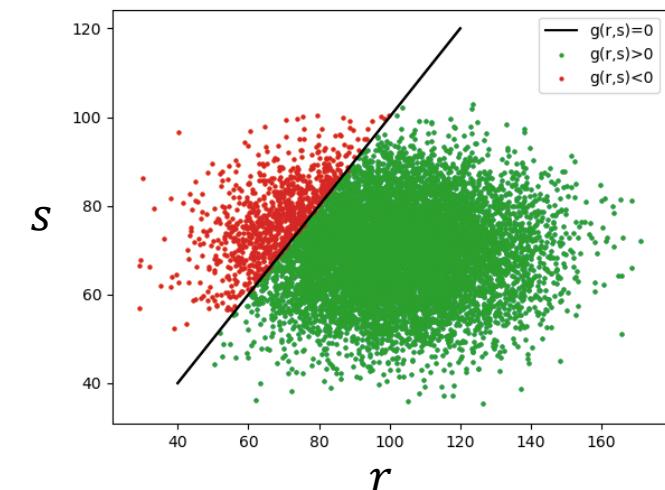
Emmanuel Ardillon, Vincent Chabridon, Bertrand Iooss (EDF)

STANDARD RELIABILITY ANALYSIS



Reliability: Does the performance of the system stay below a critical threshold z_{cr} ?

- Limit-state function: $g(\mathbf{x}, \mathbf{d}) = z_{th} - z$
- Failure domain: $D_F = \{\mathbf{x}, g(\mathbf{x}, \mathbf{d}) \leq 0\}$
- Uncertainty handled by using probabilistic theory:
 $x_i \rightarrow F_{X_i}(x_i)$
- Failure probability:
 $P_f = \Pr(g(\mathbf{X}, \mathbf{d}) \leq 0) = \int_{D_F} f(\mathbf{x}) d\mathbf{x}$



CONTEXT

- Robustness evaluation of reliability quantities (probability of failure) using non probabilistic nested convex sets
- What other uncertainty representations exist?
- Bibliographic review of uncertainty models used in structural reliability
 - How is uncertainty modeled?
 - How is uncertainty propagated through the computer model?
 - How can the results be interpreted?
 - Can different models be used together?
 - What is the effect of the uncertainty model on the robustness?

Summary

1. Uncertainty representations
2. Hybrid Reliability Analysis
3. Robust Hybrid Reliability Analysis
4. Results

UNCERTAINTY REPRESENTATIONS

- Interval model:

$$X_i \in I_{X_i} = [X_i^L, X_i^U] \rightarrow g(\mathbf{X}, \mathbf{d}) \in [Z^L, Z^U]^{(int)}$$

$$\begin{cases} Z^L = \min_{I_X} g(\mathbf{X}, \mathbf{d}) \\ Z^U = \max_{I_X} g(\mathbf{X}, \mathbf{d}) \end{cases} , \quad I_X = \times_{i=1}^{n_x} I_{X_i}$$

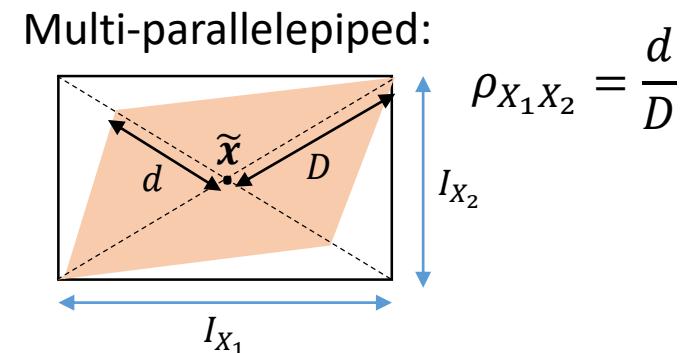
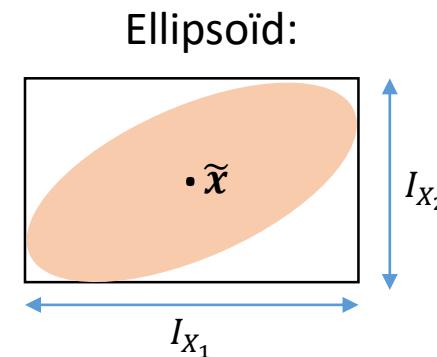
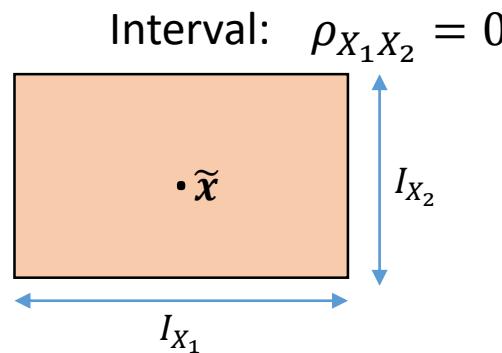
- RS example:

$$g_{RS} = g(R, S) = R - S$$

$$\begin{cases} R \in [100; 150] \\ S \in [60; 110] \end{cases} \Rightarrow g_{RS} \in [-10; 90]$$

UNCERTAINTY REPRESENTATIONS

- Convex models: $\mathbf{X} \in C(I_{\mathbf{X}}, \rho) \rightarrow g(\mathbf{X}, \mathbf{d}) \in [Z^L, Z^U]^{(int)}$



- RS example:

$$\begin{cases} C = MP \\ I_R = [100; 150] \\ I_S = [60; 110] \\ \rho_{RS} = 0.4 \end{cases} \Rightarrow g_{RS} \in [18.6; 61.4]$$

UNCERTAINTY REPRESENTATIONS

- Evidence theory:

- $\Omega(X)$: Power set
- A : Focal set of X_i
- ν_X : Basic Probability Assignment

$$\begin{aligned}\nu_X : & \Omega(X) \rightarrow [0,1], \\ & A \rightarrow \nu_X(A) ; \sum_A \nu(A) = 1\end{aligned}$$

- Two probability measures: $\text{Bel}(E) \leq \Pr(E) \leq \text{Pl}(E)$

$$\text{Bel}(E) = \sum_{A \subseteq E} \nu_X(A) ; \text{Pl}(E) = \sum_{A \cap E \neq \emptyset} m_X(A)$$

- Application to SRA: $E = \{g(\mathbf{X}, \mathbf{d}) \leq \mathbf{0}\}$

$$\sum_{G_j \subseteq]-\infty, 0]} \nu_G(G_j) \leq P_f \leq \sum_{G_j \cap]-\infty, 0] \neq \emptyset} \nu_G(G_j)$$

UNCERTAINTY REPRESENTATIONS

- RS example:

R_i	$\nu_R(R_i)$	S_i	$\nu_S(S_i)$
[100; 112.5]	0.25	[60; 72.5]	0.25
[112.5; 125]	0.25	[72.5; 85]	0.25
[125; 137.5]	0.25	[85; 97.5]	0.25
[137.5; 150]	0.25	[97.5; 110]	0.25

[27.5; 52.5]	[40; 65]	[52.5; 77.5]	[65; 90]
[15; 40]	[27.5; 52.5]	[40; 65]	[52.5; 77.5]
[2.5; 27.5]	[15; 40]	[27.5; 52.5]	[40; 65]
[-10; 15]	[2.5; 27.5]	[15; 40]	[27.5; 52.5]

$$G_j \xrightarrow{\nu_G(G_j) = \frac{1}{16}}$$

$$0 \leq P_f \leq \frac{1}{16}$$

UNCERTAINTY MODELS REVIEW

- Possibility distribution:

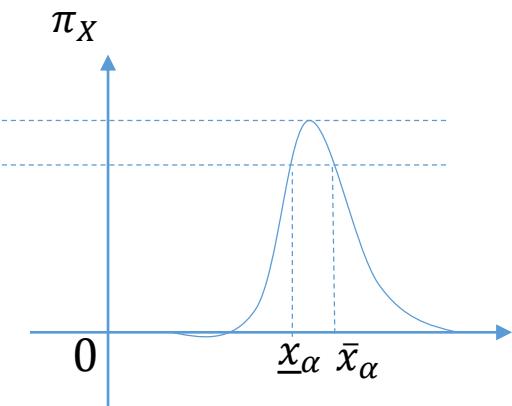
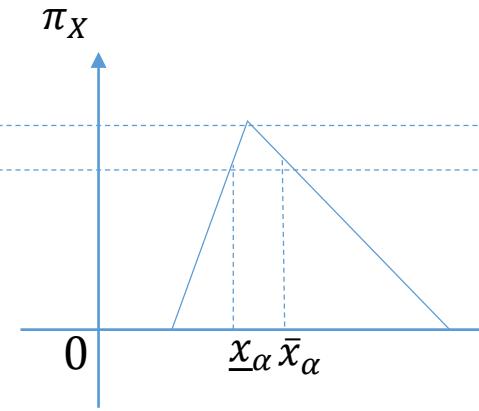
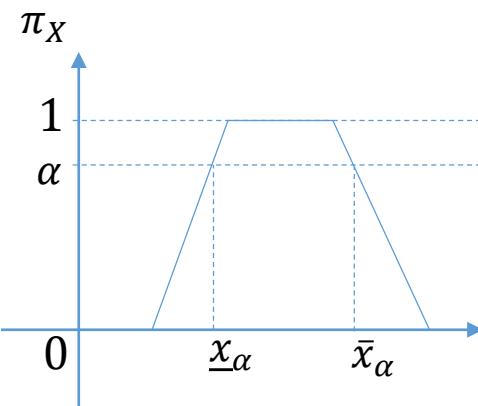
$$\begin{aligned}\pi_X: & \Omega(X) \rightarrow [0,1] \\ x & \rightarrow \pi_X(x)\end{aligned}$$

- Two measures: $\Pi(E) \leq \Pr(E) \leq N(E)$

$$\Pi_X(E) = \sup_{x \in E} \pi_X(x) ; N_X(E) = \inf_{x \notin E} (1 - \pi_X(x))$$

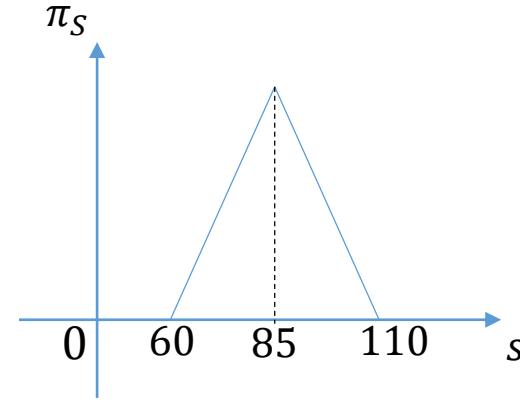
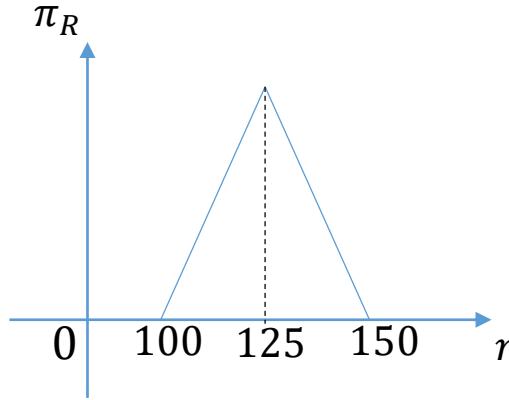
- α -cuts:

$$[\underline{x}_\alpha, \bar{x}_\alpha] = \{x, \pi_X(x) \geq \alpha\}$$



UNCERTAINTY MODELS REVIEW

- RS example:



$$\underline{P}_f \leq P_f \leq \bar{P}_f$$

$$\underline{P}_f = \Pr\left(\max_{I_{RS}(\alpha)} g_{RS} \leq 0\right) = 0$$

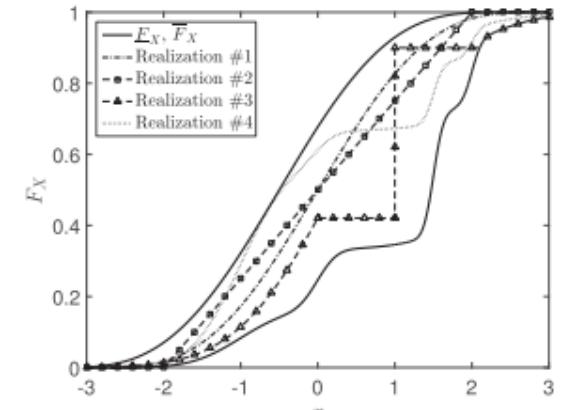
$$\bar{P}_f = \Pr\left(\min_{I_{RS}(\alpha)} g_{RS} \leq 0\right) = 0.008$$

$$I_{RS}(\alpha) = [r_{\alpha_1}, \bar{r}_{\alpha_1}] \times [s_{\alpha_2}, \bar{s}_{\alpha_2}], \alpha_1 \sim U(0,1), \alpha_2 \sim U(0,1)$$

UNCERTAINTY MODELS REVIEW

- P-box model:
 - Free p-box:

$$\bar{F}_{X_i}(x_i) \leq F_{X_i}(x_i) \leq \underline{F}_{X_i}(x_i)$$

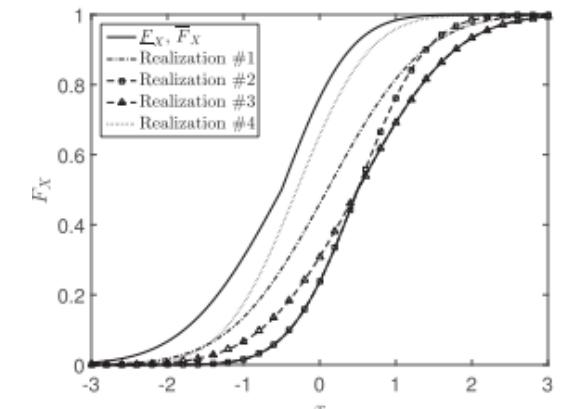


(a) Free p-box

Schöbi (2017)

- Parametric p-box:

$$X_i \rightarrow F_{X_i}(x_i, \boldsymbol{\theta}_i), \quad \boldsymbol{\theta}_i \in D_{\theta_i}$$
$$D_{\theta_i} = \left\{ [\theta_{i1}^L, \theta_{i1}^U], \dots, [\theta_{jN_{\theta_i}}^L, \theta_{jN_{\theta_i}}^U] \right\}$$



(b) Parametric p-box

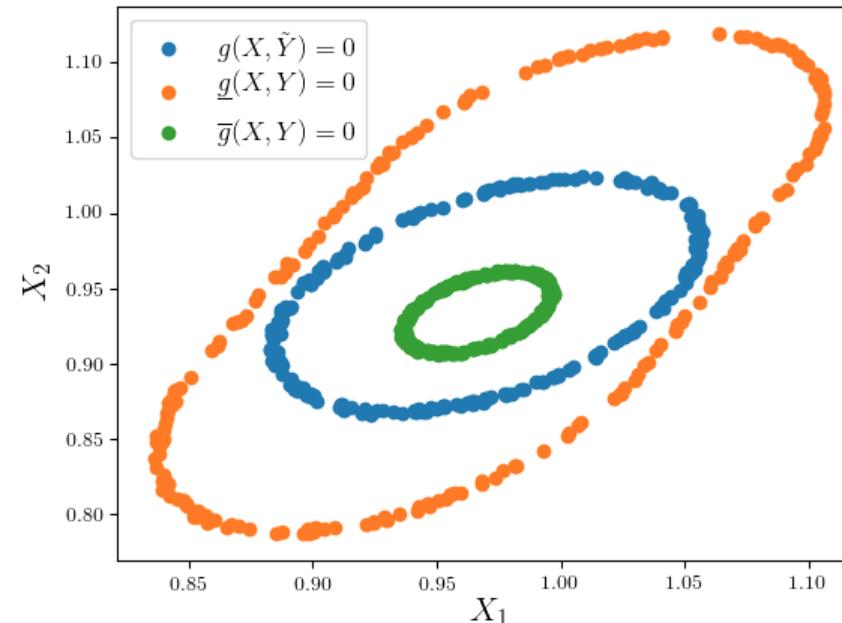
Hybrid Reliability Analysis (HRA)

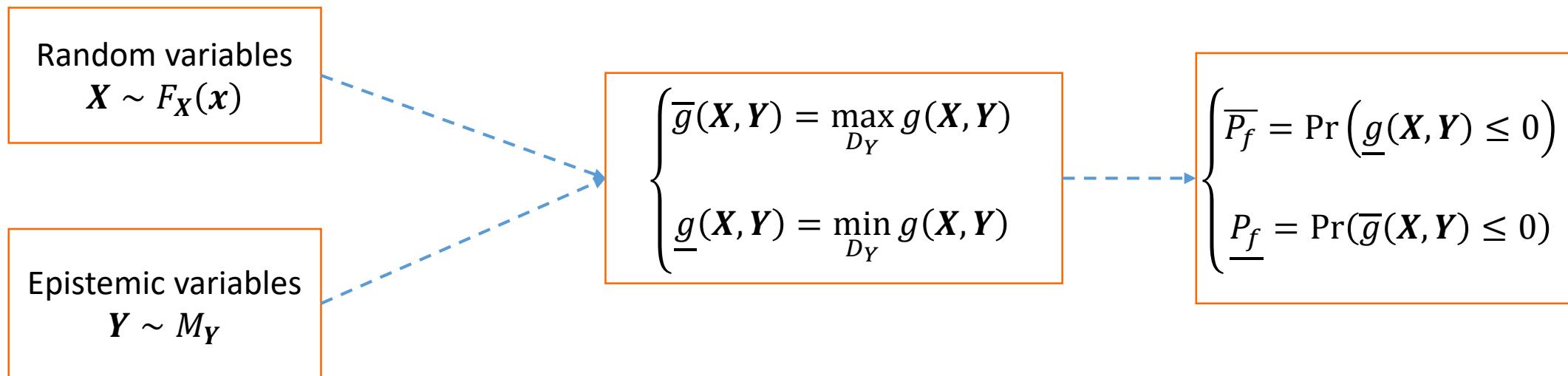
- Presence of both purely aleatory uncertainty \mathbf{X} with epistemic uncertainty \mathbf{Y} .
- Hybrid limit-state function: $g(\mathbf{X}, \mathbf{Y})$
- Example:

$$g(X_1, X_2, Y) = 100(X_2 - X_1^2)^2 + (X_1 - 1)^2 + 100(Y - X_2^2)^2 + (X_2 - 1)^2 - 3$$

- $X_1 \sim N(0; 1)$, $X_2 \sim N(0; 1)$
- $I_Y = [0.7; 1.1]$, $\tilde{Y} = 0.9$

- $$\begin{cases} \bar{g}(\mathbf{X}, \mathbf{Y}) = \max_{Y \in I_Y} g(\mathbf{X}, Y) \\ \underline{g}(\mathbf{X}, \mathbf{Y}) = \min_{Y \in I_Y} g(\mathbf{X}, Y) \end{cases}$$





How to propagate the different uncertainty representations in order to estimate the bounds on the failure probability?

- Random Sets (RS):

- A random set Γ is the function:

$$\Gamma_Y : | [0,1] \rightarrow S_Y$$

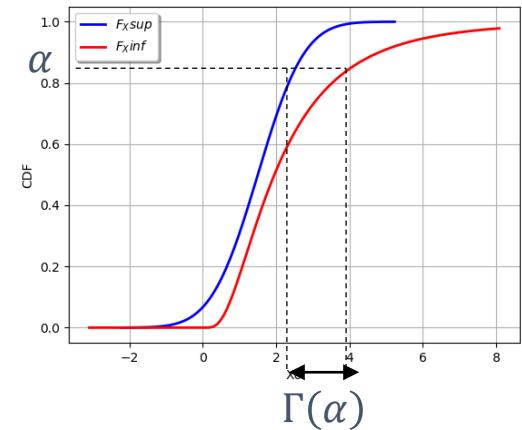
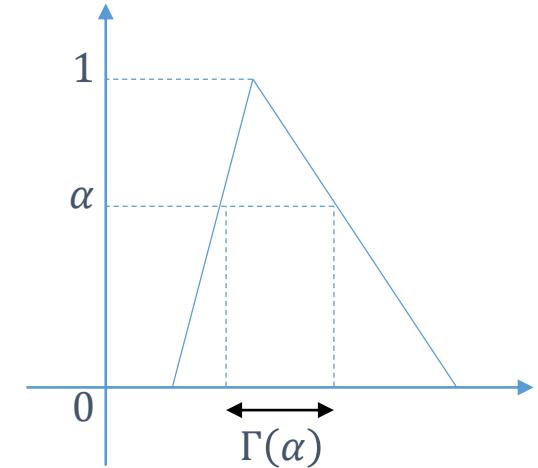
$$\alpha \rightarrow \Gamma_Y(\alpha)$$

- $\Gamma_Y(\alpha)$: focal element in the support S_Y
 - Bounds on the probability of the event E :

$$\underline{P}_\Gamma(E) = P_\Omega(\{\alpha \in [0,1] : \Gamma(\alpha) \subseteq E, \Gamma(\alpha) \neq \emptyset\})$$

$$\overline{P}_\Gamma(E) = P_\Omega(\{\alpha \in [0,1] : \Gamma(\alpha) \cap E \neq \emptyset\})$$

- RS - Interval: $\Gamma(\alpha) = Y^M, \forall \alpha \in [0,1]$
- RS - Convex: $\Gamma(\alpha) = C(Y^M, \rho), \forall \alpha \in [0,1]$
- RS - Possibility: $\Gamma(\alpha) = \{y \in S_Y : \pi(y) \geq \alpha\}, \forall \alpha \in [0,1]$
- RS - CDF: $\Gamma(\alpha) = F_Y^{-1}(\alpha), \forall \alpha \in [0,1]$
- RS - Free p-box: $\Gamma(\alpha) = [\bar{F}_Y^{-1}(\alpha), \underline{F}_Y^{-1}(\alpha)], \forall \alpha \in [0,1]$
- RS - parametric p-box \rightarrow RS - free P-box



- Sampling a random set in dimension $N > 1$:

$$\Gamma_Y(\boldsymbol{\alpha}) = \times_{i=1}^N \Gamma_{Y_i}(\alpha_i), \alpha_i \sim U(0,1)$$

- Application to our reliability problem:

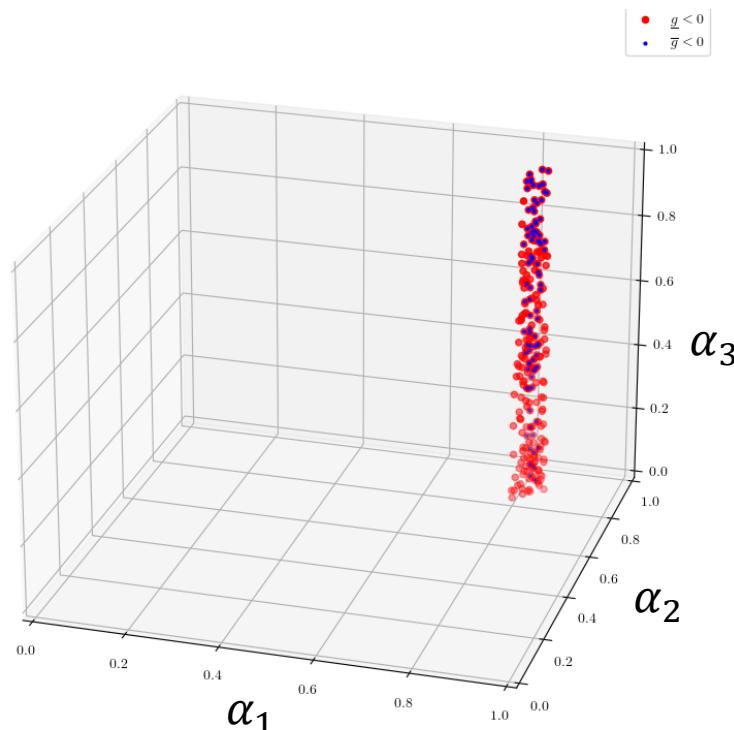
$$\begin{cases} \overline{P_f} = \Pr(\underline{g}(X, Y) \leq 0) = \Pr\left(\min_{\Gamma_{XY}(\boldsymbol{\alpha})} g(\boldsymbol{\alpha}) \leq 0\right) \\ \underline{P_f} = \Pr(\overline{g}(X, Y) \leq 0) = \Pr\left(\max_{\Gamma_{XY}(\boldsymbol{\alpha})} g(\boldsymbol{\alpha}) \leq 0\right) \end{cases} \quad \alpha_i \sim U(0,1)$$

- Hybrid problem reduced to two standard reliability analysis

- Example:

$$g(X_1, X_2, Y) = 100(X_2 - X_1^2)^2 + (X_1 - 1)^2 + 100(Y - X_2^2)^2 + (X_2 - 1)^2 - 3$$

$X_1 \sim N(0; 1)$, $X_2 \sim N(0; 1)$, $Y \sim Tri(0.765; 0.9; 1.035)$, MC sampling $N = 10^5$



$$\widehat{\overline{P}}_f = 0.00237$$

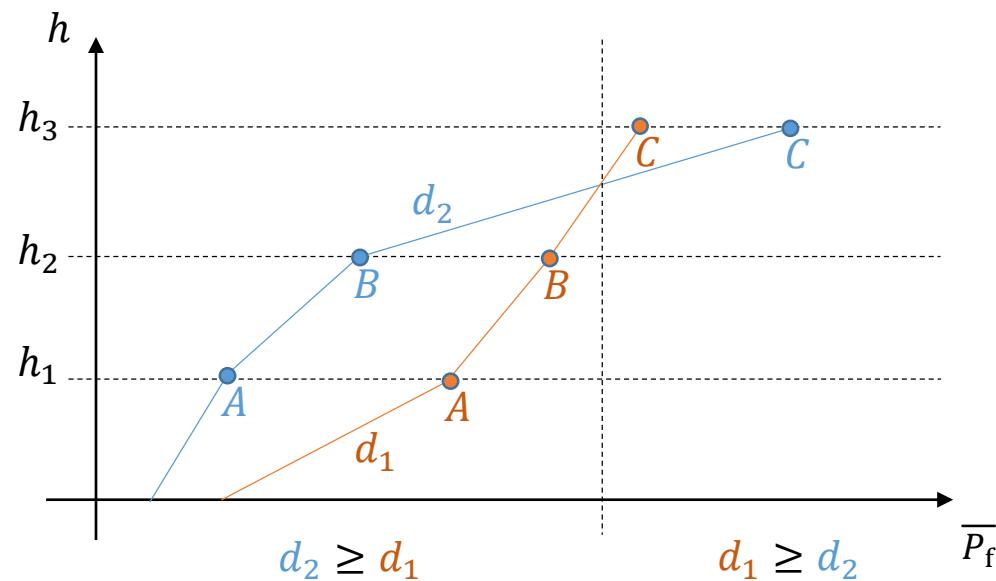
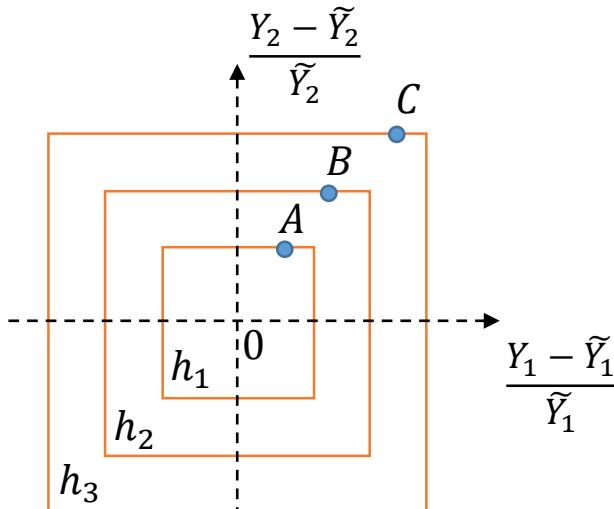
$$\underline{\widehat{P}}_f = 0.00087$$

INFO-GAP ROBUSTNESS ANALYSIS

Objective: Guarantee that a quantity of interest stays acceptable after perturbations on the nominal design.

- Info-Gap **robustness** and **opportuneness** quantification:

$$h^* = \max_h \left\{ \max_{Y \in U(h, \tilde{Y})} P_f \leq P_f^{cr} \right\}, h \geq 0 \quad ; \quad \beta^* = \min_h \left\{ \min_{Y \in U(h, \tilde{Y})} P_f \leq P_f^{rw} \right\}, h \geq 0$$



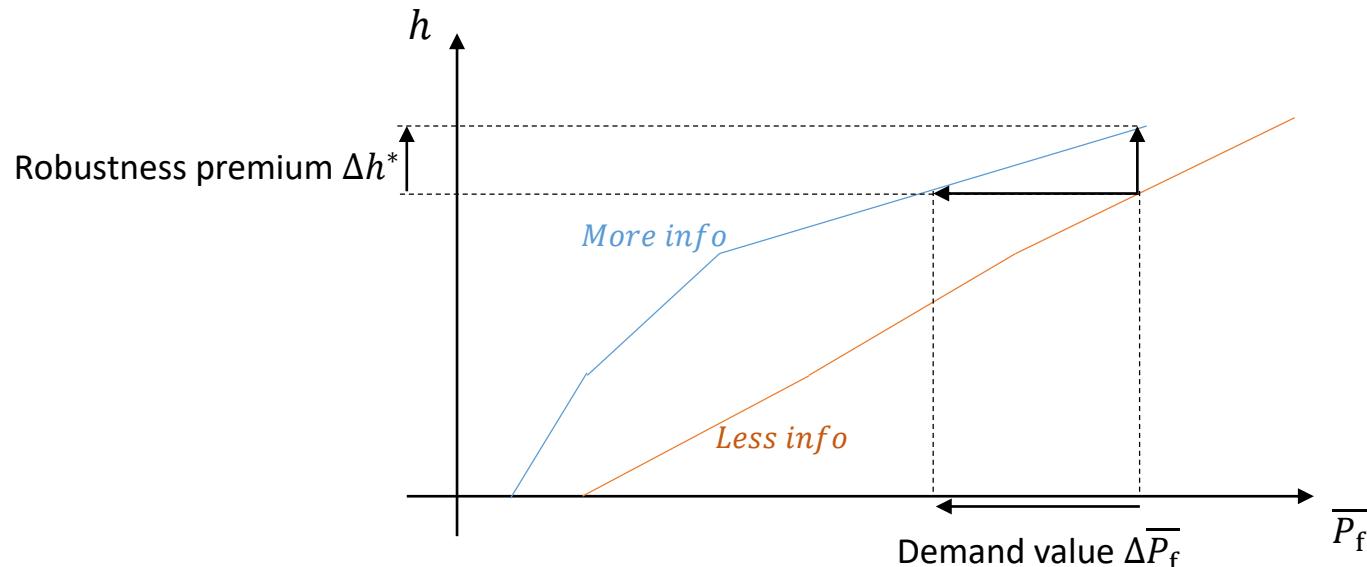
- IG Value of Information:

The uncertainty model U_1 is more informative than U_2 if:

$$U_1(h, \tilde{Y}) \subset U_2(h, \tilde{Y})$$

which implies:

$$h_2^*(p_f^{cr}) \leq h_1^*(p_f^{cr})$$

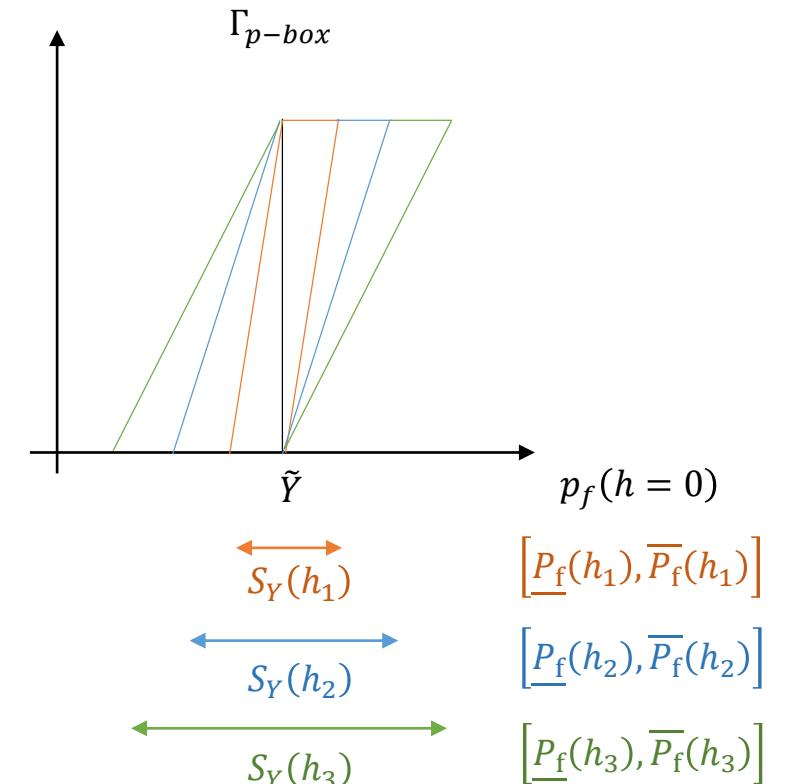
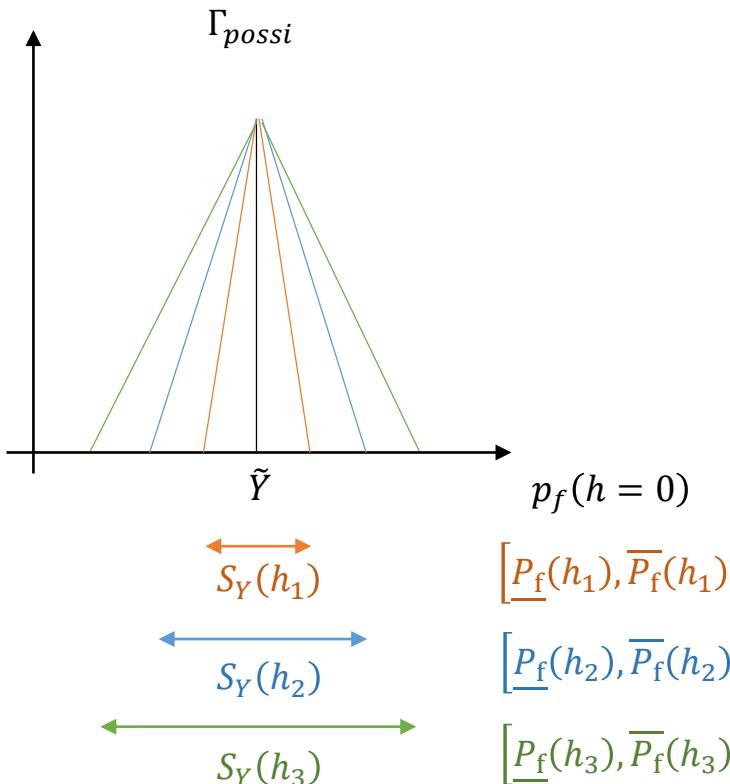
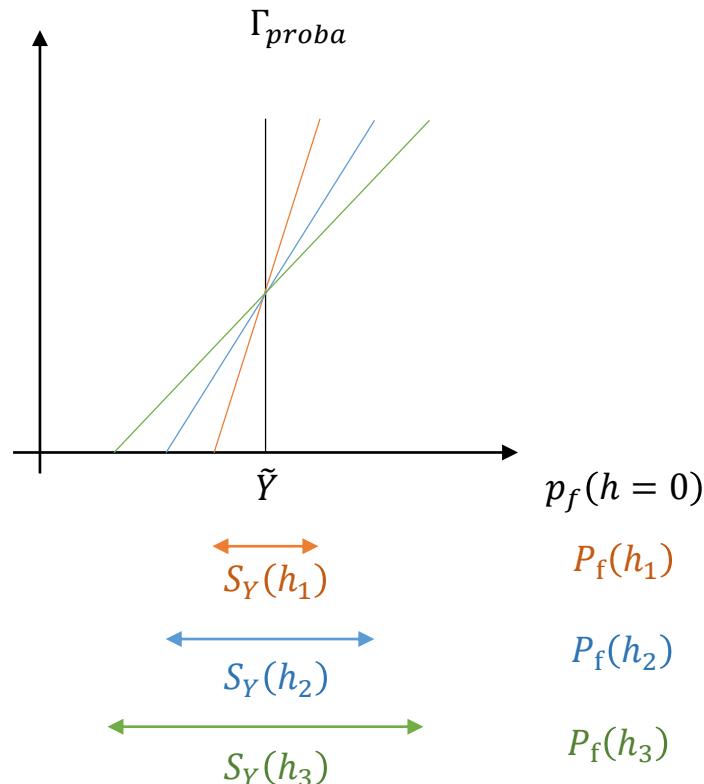


ROBUST HRA

ROBUST HRA

$$\Gamma_Y: \Omega \rightarrow S_Y, \alpha \rightarrow \Gamma_Y(\alpha)$$

$$S_Y = \{Y, \tilde{Y}(1-h) \leq Y \leq \tilde{Y}(1+h)\}$$



ROBUST HRA

- Demand value between $\Gamma_Y^{(i)}$ and $\Gamma_Y^{(j)}$:

$$r_{\max} = \Delta P_f^{(ij)} = 1 - \frac{\bar{P}_f^{(i)}}{\bar{P}_f^{(j)}}$$

- Groups of comparison:
 - G_1 : Interval – Trapeze poss – Triangle poss – Uniform
 - G_2 : free p-box – param p-box
 - G_3 : Interval – Convex parallelepiped
 - G_4 : DS – Proba and DS – Poss

ROBUST HRA

Sensitivity analysis:

$$S_{r_{max}} = \frac{\Delta P_f^{(ij)(k^*)}}{\Delta P_f^{(ij)}}$$

$$\Delta P_f^{(ij)(k^*)} = 1 - \frac{\bar{P}_f^{(i)}}{\bar{P}_f^{(j)(k^*)}}$$

$$Y^{(i)} = [Y_1^{(i)}, \dots, Y_{N_Y}^{(i)}]$$

$$Y^{(j)(k^*)} = [Y_1^{(i)}, \dots, Y_{k^*}^{(j)}, \dots, Y_{N_Y}^{(i)}]$$

TOY CASE 1: MATHEMATICAL FUNCTION

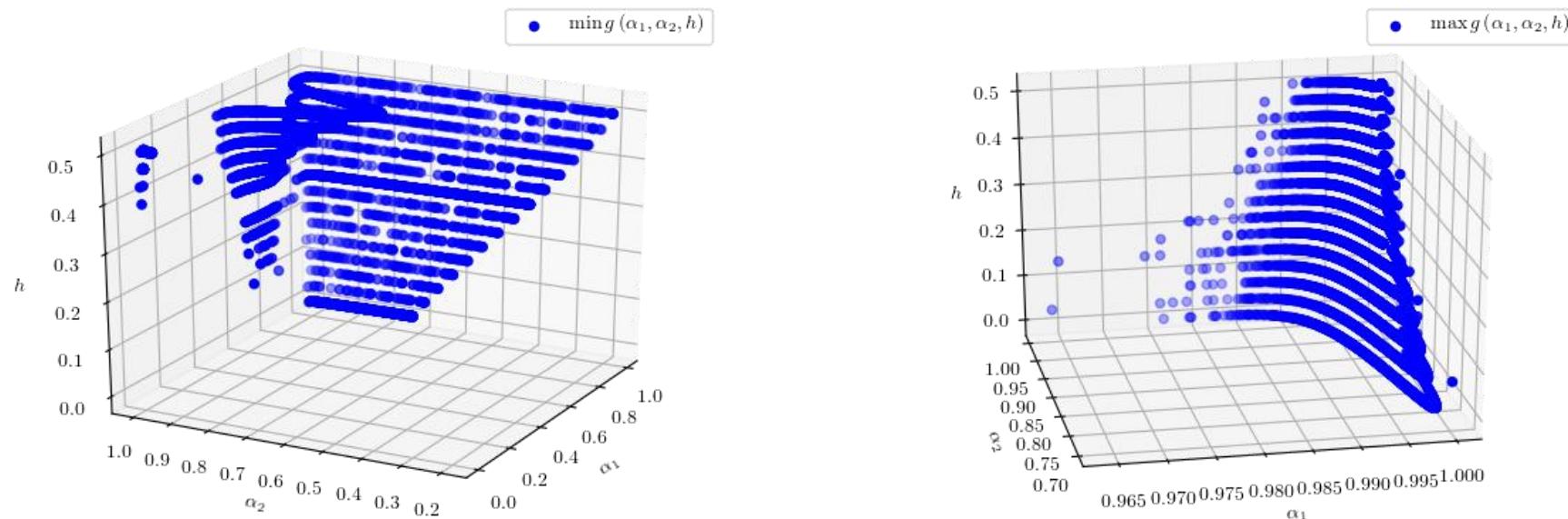
TC1: MATHEMATICAL FUNCTION

- Hybrid limit-state:

$$g(X, Y) = Y + \sin\left(\frac{5(X_1 + 1.5)}{2}\right) - \frac{((X_1 + 1.5)^2 + 4). (X_2 + 1.5)}{20}$$

- $X_1, X_2 \sim N(0,1)$, $\tilde{Y} = 3.5$, $h \in [0, 0.5]$

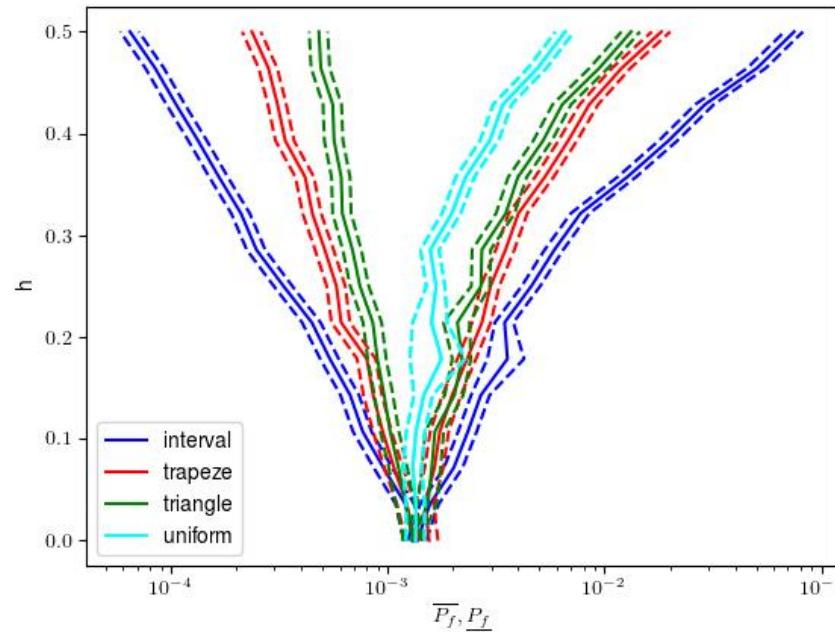
- Hybrid limit-state shape:



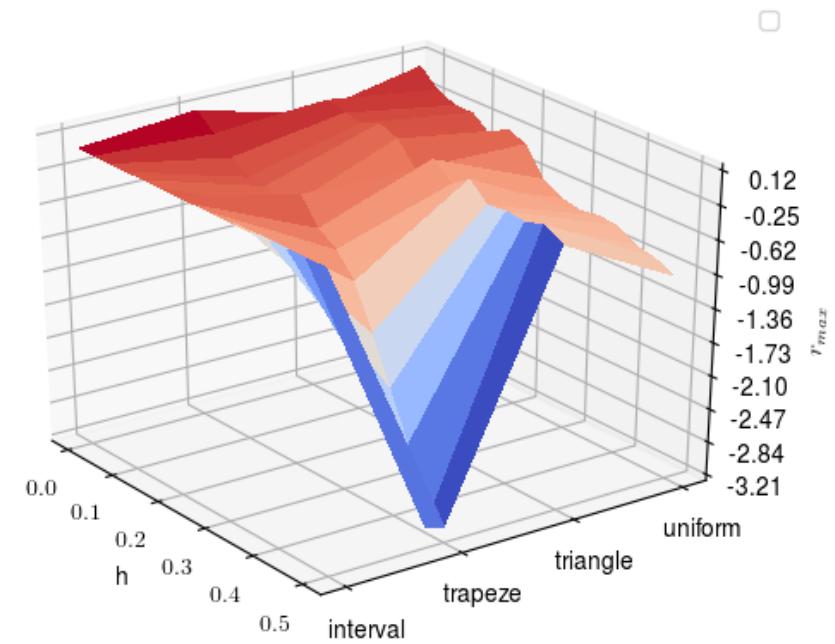
TC1: MATHEMATICAL FUNCTION

- Results for G_1 :

Robustness and opportunity

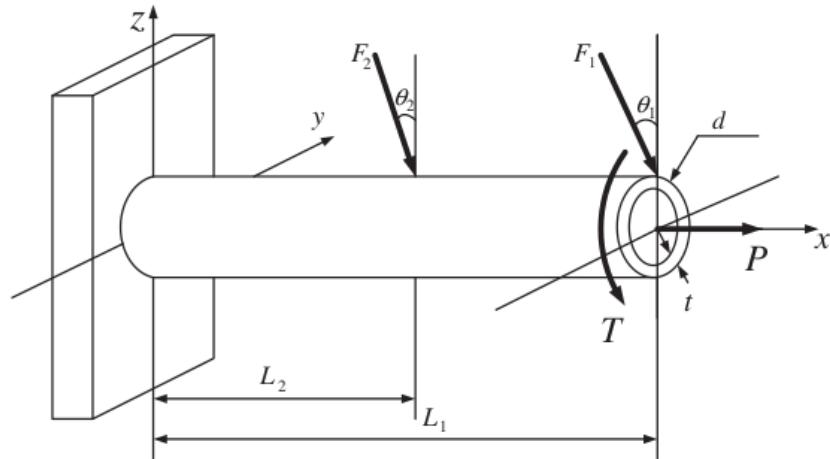


Robustness premium



TOY CASE 2: CANTILEVER BEAM

TC2: CANTILEVER BEAM



Y	\tilde{Y}
$F_1(kN)$	3
$F_2(kN)$	3
$\theta_1(rad)$	0.175
$\theta_2(rad)$	0.350
$T(N.m)$	90

$$g(X, Y) = \sigma_y - \sqrt{\sigma_x^2 + 3\tau_{xz}^2}$$

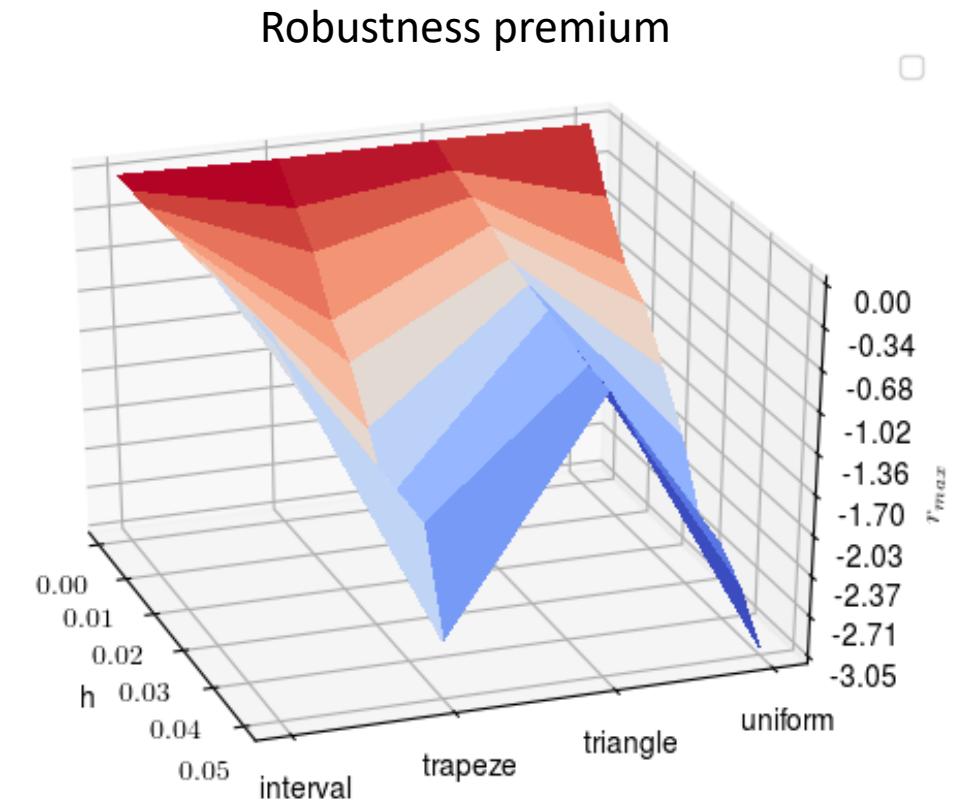
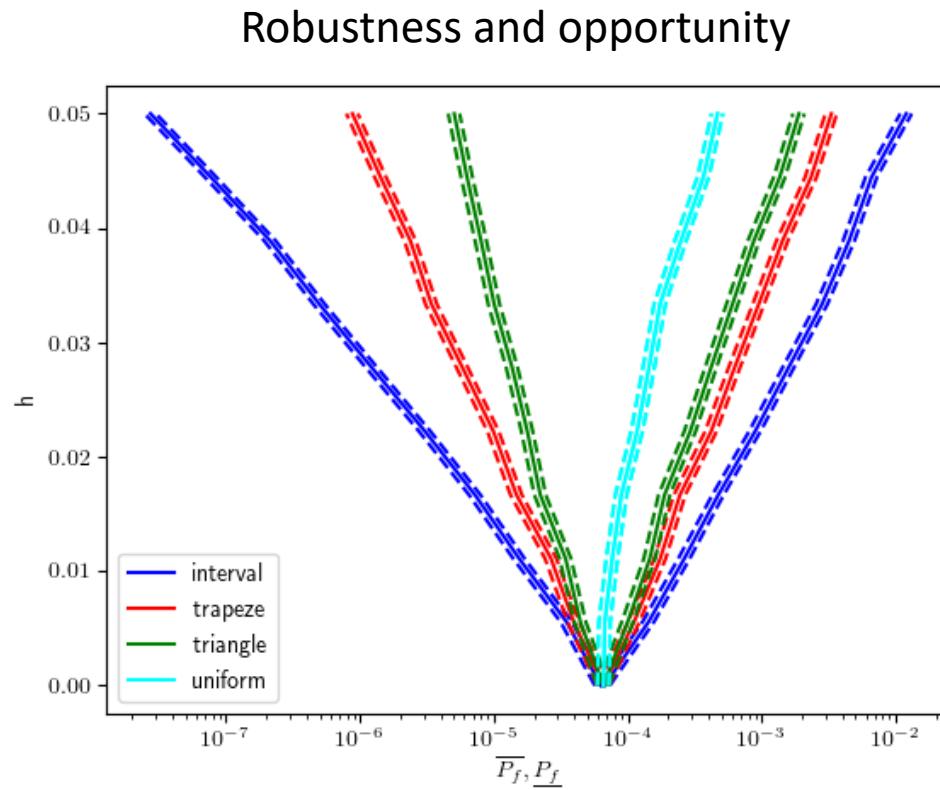
$$\sigma_x = \frac{P + F_1 \sin \theta_1 + F_2 \sin \theta_2}{A} + \frac{Md}{2I}$$

$$\tau_{xz} = \frac{Td}{4I}$$

X	Distribution	Param 1	Param 2
$P(kN)$	Normal	12	1.2
$t(mm)$	Normal	5	0.1
$d(mm)$	Normal	42	0.5
$L_1(mm)$	Uniform	119	121
$L_2(mm)$	Uniform	59	60

TC2: CANTILEVER BEAM

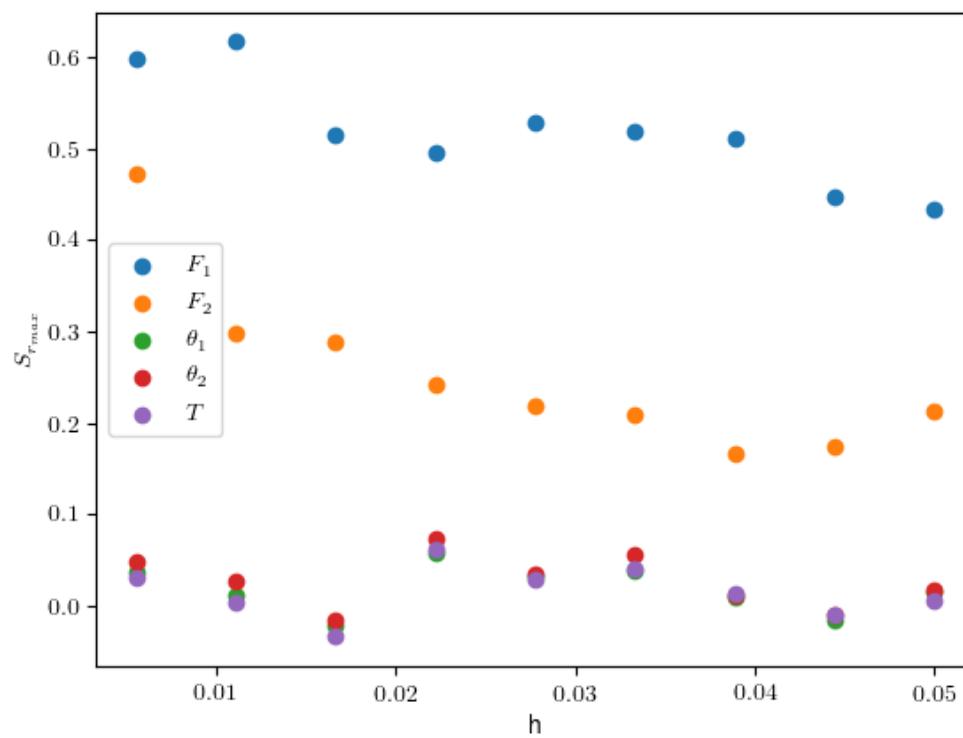
- Results G_1 :



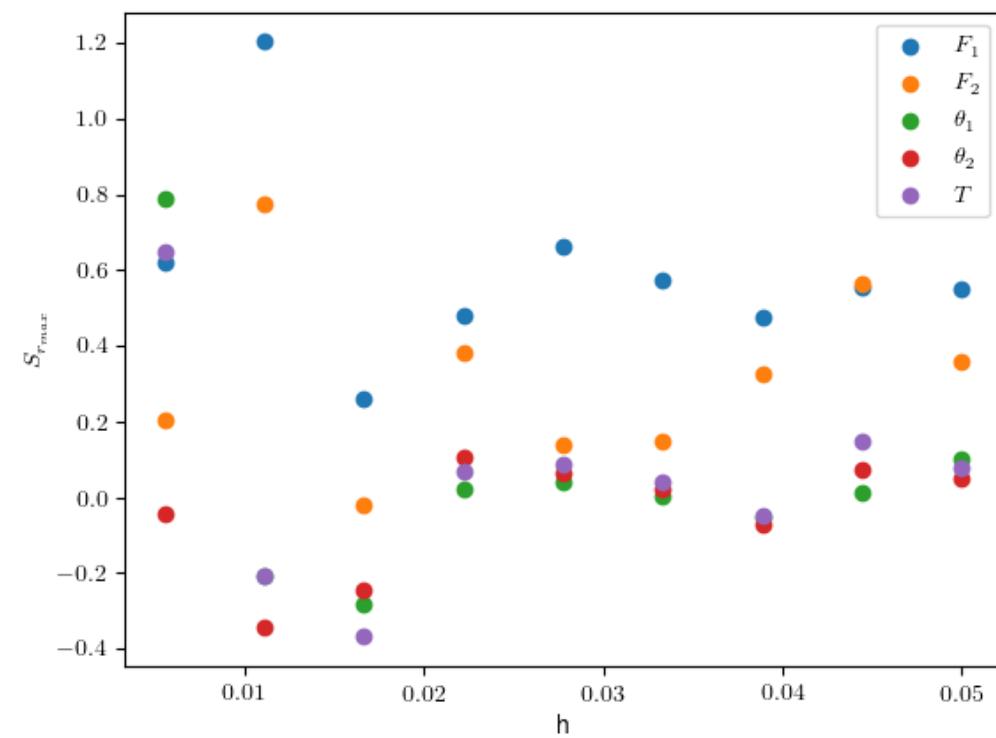
TC2: CANTILEVER BEAM

- Sensitivity Results G_1 :

Interval to trapezoidal

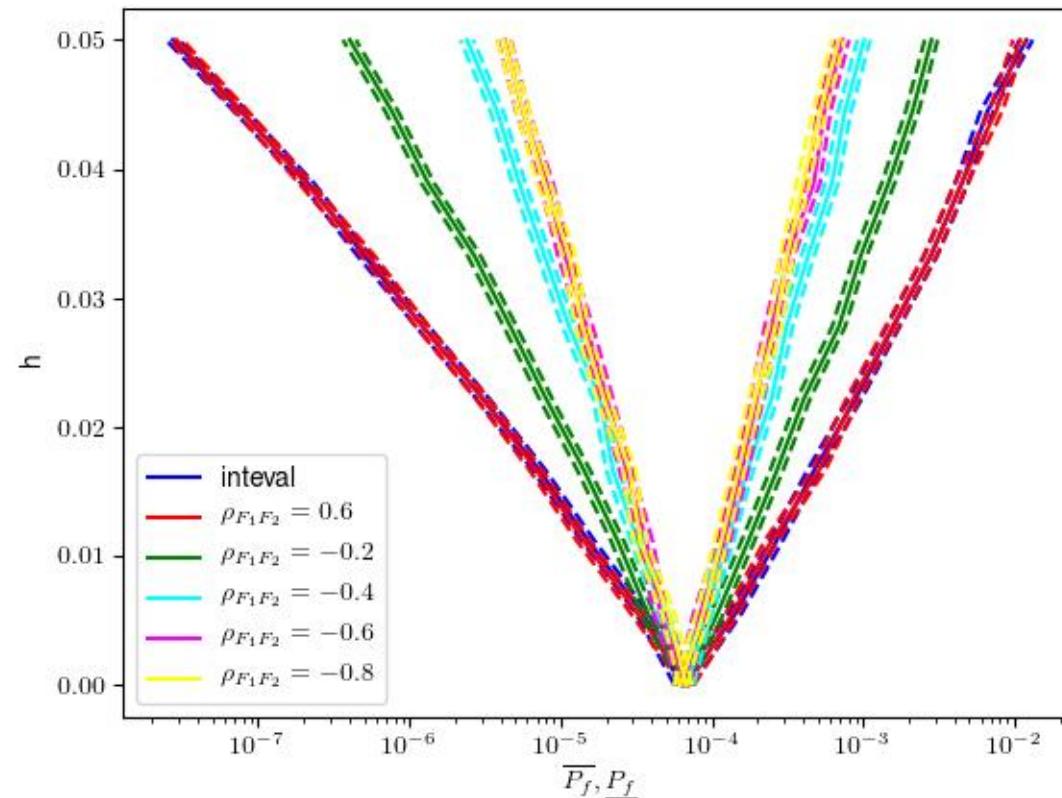


Trapezoidal to triangular



TC2: CANTILEVER BEAM

- Results G_3 :



CONCLUSIONS AND PERSPECTIVES

Conclusions:

- Mixed uncertainty is often present in industrial applications
- Random Sets is a suitable framework for modeling and propagating different types of uncertainty models
- An info-gap approach is proposed to compare the different epistemic models through a robustness analysis

Perspectives:

- How to quantify the value of information?
- Sensitivity analysis, dependency between variables with different uncertainty representations
- Very expensive calculations → Need for smart optimization algorithms and surrogate models

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